Integrability of Models Arising from Motions of Plane Curves

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Four sets of nonlinear evolution equations which are viewed as generalizations of $K(n,m)$ ($n = m+1, m+2$) models, are introduced. It is shown that these equations, together with the $K(n,m)$ ($n = m+1, m+2$) models, arise naturally from motions of curves in several geometries, and they are Painlevé integrable.

Other gauge equivalent integrable equations are obtained by use of the equivalence between integrable equations for the curvature and graph of the curves. In particular, we obtain the generalized WKI equation and its one-loop soliton solutions.

Key words: Motion of Plane Curve, Integrable Equation; Loop-soliton; Painlevé Property; Gauge Transformation.

1. Introduction

To investigate the role played by the nonlinear dispersion in pattern formation, Rosenau and Hyman [1, 2] have proposed $K(n,m)$ models of the form

$$\phi_t \pm (\phi^n)_\sigma + (\phi^m)_{\sigma\sigma\sigma} = 0, \tag{1}$$

where m and n are integers, ϕ is a smooth function of the time t and the space variable σ. In particular, the $K(2,1)$ and $K(3,1)$ models are the well-known KdV and mKdV equations. A remarkable property to the $K(m,m)$ model is that it has solitons with compact support [1]. Recently, Lou and Wu [3] proved that the $K(m+1,m)$ and $K(m+2,m)$ models are Painlevé integrable. So $K(m+1,m)$ and $K(m+2,m)$ models should have some features of integrable equations.

It has been known for a long time that integrable equations are closely related to motion of curves or surfaces. This relationship provides new insight and structures of integrable equations. The pioneering work is due to Hasimoto [4]. He showed that the nonlinear Schrödinger equation arises from the dynamics of a non-stretching string in Euclidean space. Since then, much work on this topic has been done. It is known that many integrable equations arise naturally from motions of plane or space curves in Euclidean or other geometries [4 – 19]. In particular, the mKdV and KdV equations arise from motions of plane curves respectively in Euclidean and centro-affien geometries [4, 17]. It is natural to ask whether the $K(m+1,m)$ and $K(m+2,m)$ models can be obtained from motions of plane curves in certain geometries.

The equivalence between integrable equations for the curvature and invariant motion leads to new integrable equations. Since very often one can express a motion law as a single evolution equation for some quantity, in view of this equivalence this evolution equation should also be integrable. In general, there are many ways to reduce the motion to a single equation. As an illustration we recall that the curvature of an inextensible plane curve γ satisfies the mKdV hierarchy [14]. We suppose this flow can be expressed as the graph $(x, u(x,t))$ of some function u on the x-axis. Using the fact that the normal speed of the curve γ, $u_t/(1+u_x^2)^{1/2}$, is given by $-k_s$, one finds that u satisfies

$$u_t = \left[\frac{u_{xx}}{(1+u_x^2)^{3/2}}\right]_x. \tag{2}$$

This is nothing but the well-known WKI equation. In fact, it turns out that the integrability of (2) was established by Wadati, Konno, and Ichikawa [20], who showed that it is the compatibility condition of a certain WKI-scheme of inverse scattering transformation. This WKI-scheme for u is connected to the AKNS-scheme for k by a gauge transformation, explicitly displayed in [21] (see also [22]). Thus this approach gives a geometric interpretation of this correspondence.

In this paper, we will introduce the $K(n,m,1)$ $(n=m+1,m+2)$ models

$$\phi_t+(\phi^n)_\sigma+(\phi^m)_{\sigma\sigma\sigma}+a\phi_\sigma=0 \tag{3}$$

and the $K(n,m,m)(n=m+1,m+2)$ models

$$\phi_t+(\phi^n)_\sigma+(\phi^m)_{\sigma\sigma\sigma}+a(\phi^m)_\sigma=0 \tag{4}$$

by investigating motions of curves in some geometries, where ϕ and σ denote, respectively, the curvature and arc-length of the geometries, and a is a constant. In particular, the $K(2,1,1)$ and $K(3,1,1)$ models are integrable and can be solved via the inverse scattering method. We will show that these models and $K(n,m)(n=m+1,m+2)$ models arise naturally from motions of curves in Euclidean, centro-affine, $S^2(R)$ and fully affine geometries. The Painlevé integrability of these models will be investigated. The outline of this paper is as follows. In Sect. 2 we show how to derive these models from motions of curves in several geometries. The Painelevé integrability of these models is investigated in Section 3. In Sect. 4, some gauge equivalent integrable equations are presented. In particular, the equation which is gauge equivalent to the $K(m+2,m)$ model is the generalized WKI equation [2]

$$u_t=\left[\left(\frac{u_{xx}}{(1+u_x^2)^{3/2}}\right)^m\right]_x. \tag{5}$$

Its one-loop soliton solutions are obtained in Section 5. Section 6 gives concluding remarks on this work.

2. Derivation of the Models

2.1. $K(m+2,m,1)$ *Model*

Recall that the Euclidean motion of plane curves is of the form [14, 15]

$$\gamma_t=f\boldsymbol{n}+g\boldsymbol{t}, \tag{6}$$

where f and g, depending on the curvature k of curves and its derivatives with respect to the arc-length s, are respectively the normal and tangent velocities. $\boldsymbol{t}$ and $\boldsymbol{n}$ are, respectively, the Euclidean tangent and normal. They satisfy the Serret-Frenet formulas

$$\boldsymbol{t}_s=\kappa\boldsymbol{n},\ \boldsymbol{n}_s=-\kappa\boldsymbol{t}, \tag{7}$$

where

$$\kappa=\frac{x_py_{pp}-y_px_{pp}}{(x_p^2+y_p^2)^{\frac{3}{2}}}$$

and

$$\mathrm{d}\,s=\sqrt{x_p^2+y_p^2}\,\mathrm{d}\,p$$

are respectively the Euclidean curvature and the arc-length of a curve $\gamma=(x(p,t),y(p,t))$, where p is an arbitrary parameter. With use of (6) and (7) one computes [14, 15]

$$s_t=s(g_s-\kappa f).$$

From this we obtain the first variation formula for the perimeter $L=\oint\mathrm{d}s$ of a closed curve

$$\frac{\partial L}{\partial t}=\oint(g_s-\kappa f)\mathrm{d}s.$$

Then we have for non-stretching curves

$$g_s=\kappa f, \tag{8a}$$

and for closed curves

$$\oint\kappa f\mathrm{d}s=0. \tag{8b}$$

The time evolution for the frame is

$$\begin{pmatrix}\boldsymbol{t}\\ \boldsymbol{n}\end{pmatrix}_t=\begin{pmatrix}0 & f_s+\kappa g\\ -f_s-\kappa g & 0\end{pmatrix}\begin{pmatrix}\boldsymbol{t}\\ \boldsymbol{n}\end{pmatrix}. \tag{9}$$

The compatibility condition between (7) and (9) gives the evolution for the curvature

$$\kappa_t=f_{ss}+\kappa_sg+\kappa^2f. \tag{10}$$

Choosing $f=-(\kappa^m)_s$ and $g=-\frac{m}{m+1}\kappa^{m+1}-a,\ a=$ const, so that (8) holds, we obtain the $K(m+2,m,1)$ model

$$\kappa_t+(\kappa^m)_{sss}+\frac{m}{m+1}(\kappa^{m+2})_s+a\kappa_s=0. \tag{11}$$

For $a=0$ it is just the $K(m+2,m)$ model [2]. We will see later that it is Painlevé integrable. Therefore the $K(m+2,m)$ model arises naturally from the motion of non-stretching plane curves in Euclidean geometry. It is easy to see from the above that the $K(m+2,m,1)$ model is obtained from the AKNS-scheme without spectral parameter.

2.2. $K(m+1,m,1)$ Model

In the centro-affine geometry, the curvature and arc-length element are given, respectively, by [17]

$$\phi = \kappa h^{-3},$$

and

$$\mathrm{d}\tilde{s} = h\mathrm{d}s,$$

where $h = -<\gamma, \boldsymbol{n}>$ is the support function of a curve, $\boldsymbol{n}$ is the Euclidean normal of the curve γ, $<\cdot,\cdot>$ is the standard inner product in $\mathbb{R}^2$, κ and s are, respectively, the Euclidean curvature and arc-length of the curve. The centro-affine tangent and normal are given, respectively, by $\boldsymbol{T} = \gamma_{\tilde{s}}, \boldsymbol{N} = \gamma_{\tilde{s}\tilde{s}}/\phi$. They satisfy the Serret-Frenet formula

$$\begin{pmatrix} \boldsymbol{T} \\ \boldsymbol{N} \end{pmatrix}_{\tilde{s}} = \begin{pmatrix} 0 & \phi \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{T} \\ \boldsymbol{N} \end{pmatrix}. \tag{12}$$

We represent a motion of plane curves in centro-affine geometry in the form

$$\gamma_t = f\boldsymbol{N} + g\boldsymbol{T} \tag{13}$$

where f and g, depending on the centro-affine curvature ϕ and its derivatives with respect to the arc-length $\tilde{s}$, are respectively the normal and tangent velocities. One computes the first variation for the perimeter $L = \oint \mathrm{d}\tilde{s}$ if the curves evolve according to (13)

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \oint (g_{\tilde{s}} - 2f)\mathrm{d}\tilde{s}.$$

Similar to the Euclidean case, we require

$$g_{\tilde{s}} = 2f, \ \oint f\,\mathrm{d}\tilde{s} = 0. \tag{14}$$

Also we have the time evolution for the frame

$$\begin{pmatrix} \boldsymbol{T} \\ \boldsymbol{N} \end{pmatrix}_{t} = \begin{pmatrix} f & f_{\tilde{s}} + \phi g \\ -g & -f \end{pmatrix} \begin{pmatrix} \boldsymbol{T} \\ \boldsymbol{N} \end{pmatrix}. \tag{15}$$

The compatibility condition between (12) and (15) leads to

$$\phi_t = f_{\tilde{s}\tilde{s}} + 4\phi f + \phi_{\tilde{s}} g. \tag{16}$$

Setting $f = -(\phi^m)_{\tilde{s}}$ and $g = -2\phi^m - a$ in (16), so that (14) holds, we have the $K(m+1,m,1)$ model

$$\phi_t + (\phi^m)_{\tilde{s}\tilde{s}\tilde{s}} + \frac{4m+2}{m+1}(\phi^{m+1})_{\tilde{s}} + a\phi_{\tilde{s}} = 0. \tag{17}$$

For $a = 0$ it reduces to the $K(m+1,m)$ model [2]. Furthermore, if $m = -1/2$, it becomes the Harry Dym equation

$$\phi_t + (\phi^{-\frac{1}{2}})_{\tilde{s}\tilde{s}\tilde{s}} = 0. \tag{18}$$

Therefore the $K(m+1,m,1)$, $K(m+1,m)$ models and the Harry Dym equation arise naturally from the motions of non-stretching plane curves in the centro-affine geometry.

2.3. $K(m+2,m,m)$ Model

Doliwa and Santini [8] have discussed the motion of curves in $S^2(R)$. The motion is described by

$$\gamma_t = f\hat{\boldsymbol{n}} + g\hat{\boldsymbol{t}},$$

where $\hat{\boldsymbol{t}}$ and $\hat{\boldsymbol{n}}$ are, respectively, the tangent and normal vectors of curves on $S^2(R)$. f and g, depending on the geodesic curvature of curves on $S^2(R)$ and its derivatives with respect to the arc-length σ, are respectively the normal and tangent velocities. If the curves are non-stretching, they obey the equation for the geodesic curvature k

$$k_t = f_{\sigma\sigma} + k^2 f + k_\sigma g + \lambda^2 f, \quad \lambda = 1/R \tag{19}$$

from the compatibility condition between the Serret-Frenet formulas in $S^2(R)$

$$\begin{pmatrix} \hat{\gamma} \\ \hat{\boldsymbol{t}} \\ \hat{\boldsymbol{n}} \end{pmatrix}_{\sigma} = \begin{pmatrix} 0 & \lambda & 0 \\ -\lambda & 0 & k \\ 0 & -k & 0 \end{pmatrix} \begin{pmatrix} \hat{\gamma} \\ \hat{\boldsymbol{t}} \\ \hat{\boldsymbol{n}} \end{pmatrix},$$

and the time evolution for the frame

$$\begin{pmatrix} \hat{\gamma} \\ \hat{\boldsymbol{t}} \\ \hat{\boldsymbol{n}} \end{pmatrix}_{t} = \begin{pmatrix} 0 & \lambda g & \lambda f \\ -\lambda g & 0 & f_\sigma + kg \\ -\lambda f & -f_\sigma - kg & 0 \end{pmatrix} \begin{pmatrix} \hat{\gamma} \\ \hat{\boldsymbol{t}} \\ \hat{\boldsymbol{n}} \end{pmatrix},$$

where $\hat{\gamma} = \lambda\gamma$, and g satisfies $g_\sigma = kf$. Choosing $f = -(k^m)_\sigma$ and $g = -m/(m+1)k^{m+1}$ in (19), we obtain the $K(m+2,m,m)$ model

$$k_t + (k^m)_{\sigma\sigma\sigma} + \frac{m}{m+1}(k^{m+2})_\sigma + \lambda^2 (k^m)_\sigma = 0. \tag{20}$$

Notice that the model (20) depends on the radius of the sphere. An approach of generating integrable equations which don't depend on the radius was proposed in [8]. We point out that the $K(m+2,m,m)$ model can also be derived from the motion of plane curves in the restricted conformal geometry [19].

2.4. Defocusing $K(m+2,m,m)$ Model

The motion of plane curves in the fully affine geometry has been discussed in [18]. The arc-length and curvature in the fully affine geometry are, respectively, given by

$$\mathrm{d}\,l=\mu^{\frac{1}{2}}\kappa^{\frac{1}{3}}\,\mathrm{d}\,s,$$

and

$$\phi=\mu^{-1}\mu_l,$$

where $\mu=\kappa^{\frac{4}{3}}+\frac{1}{3}(\kappa^{-\frac{5}{3}}\kappa_s)_s$, κ and s are, respectively, the Euclidean curvature and arc-length. The fully affine tangent and normal are defined by $\boldsymbol{T}=\gamma_l$, and $\boldsymbol{N}=\gamma_{ll}$. They are related to the Euclidean tangent and normal by

$$\boldsymbol{T}=\mu^{-\frac{1}{2}}\kappa^{-\frac{1}{3}}\boldsymbol{t},$$
$$\boldsymbol{N}=\mu^{-1}\kappa^{\frac{1}{3}}\boldsymbol{n}+\mu^{-\frac{1}{2}}\kappa^{-\frac{1}{3}}(\mu^{-\frac{1}{2}}\kappa^{-\frac{1}{3}})_s\boldsymbol{t}.$$

This allows one to write the Serret-Frenet formula in the fully affine geometry

$$\begin{pmatrix}\boldsymbol{T}\\ \boldsymbol{N}\end{pmatrix}_l=\begin{pmatrix}0 & 1\\ -\frac{1}{2}(\phi^2+\phi_l+\frac{2}{9}) & -\frac{3}{2}\phi\end{pmatrix}\begin{pmatrix}\boldsymbol{T}\\ \boldsymbol{N}\end{pmatrix}. \quad (21)$$

Now, the motion of plane curves in the fully affine geometry is specified by (13), where f and g depend on the fully affine curvature and its derivatives with respect to the arc-length l. By a straightforward computation, one obtains the first variation for the perimeter $L=\oint\mathrm{d}\,l$:

$$\frac{\mathrm{d}\,L}{\mathrm{d}t}=\oint(B_l+\frac{3}{2}A_{ll}+\frac{3}{4}\phi A_l)\mathrm{d}\,l,$$

where

$$B=g+f_l-\frac{3}{2}\phi f,$$
$$A=(f_l-\frac{3}{2}\phi f)_l+\frac{1}{2}(\phi^2+\phi_l+\frac{2}{9})f.$$

Similar to previous cases, for non-stretching curves we require

$$\oint\kappa A_l\mathrm{d}l=0,\ B=-\frac{3}{2}A_l-\frac{3}{4}\partial_l^{-1}(\phi A_l).$$

Using this, a straightforward but cumbersome computation gives the equation for the curvature [18]

$$\phi_t=3[D_l^2-\frac{1}{4}\phi^2-\frac{1}{4}\phi_l\partial_l^{-1}\phi+\frac{4}{9}]A_l. \quad (22)$$

Setting $A=-\frac{1}{3}\phi^m$ in (22), we get the defocusing $K(m+2,m,m)$ model

$$\phi_t+(\phi^m)_{lll}-\frac{m}{4(m+1)}(\phi^{m+2})_l+\frac{4}{9}(\phi^m)_l=0. \quad (23)$$

Noticing the expression for A, we find that the defocusing $K(m+2,m,m)$ models and then the defocusing mKdV equation come from the nonlocal motions of plane curves in the fully affine geometry. We point out that the defocusing $K(m+2,m,m)$ model (23) also appeared in the $SL'(2)$ geometry [19].

3. Painlevé Integrability

An equation is Painlevé integrable if its solutions are singular-valued about an arbitrary singularity manifold. Almost integrable equations which admit Lax-pair, Bäcklund transformation, and an infinite number of symmetries have been shown to have the Painlevé property [23]. Recently it was shown by Lou and Wu [3] that the $K(m+1,m)$ and $K(m+2,m)$ models are Painlevé integrable. In this section, we show that the models $K(n,m,1)$ and $K(n,m,m)$ $(n=m+1,m+2)$ are also Painlevé integrable.

To show that the models are Painlevé integrable, we first expand the solution about a singular manifold $\chi(s,t)=0$ as

$$\phi=\sum_{j=0}^{\infty}\phi_j\chi^{j+\alpha}. \quad (24)$$

According to Weiss, Tabor and Carnevale [24], substituting (24) into the models leads to conditions on α and a recursion relation for the functions ϕ_j, if α is a negative integer and the recursion relation should be consistent, then we say the models are Painlevé integrable. Substituting $\phi\sim\phi_0\chi^\alpha$ into the models, the leading order analysis implies

$$\alpha=\begin{cases}-2, & \text{for } K(m+1,m,n)\ (n=1,m) \text{ models},\\ -1, & \text{for } K(m+2,m,n)\ (n=1,m) \text{ models}.\end{cases}$$

3.1. $K(m+1,m,n)$ $(n=1,m)$ Models

The $K(m+1,m,1)$ model we consider is of the form (3) with $n=m+1$. It is just the $K(m+1,m)$

model as $a = 0$, and its Painlevé integrability is known [3]. For $m = 1$, it is a variable form of the KdV equation. To show that it is Painlevé integrable, substituting (24) with $\alpha = -2$ into the $K(m+1, m, 1)$ model, we obtain the recursion relation for the coefficients ϕ_j,

$$(j+1)(j-2m)(j-4m+2)\phi_j = f_j(\chi_\sigma, \chi_t, \cdots, \phi_0, \phi_1, \phi_2, \cdots, \phi_{j-1}).$$

The resonances occur at $j = -1, 2m, 4m-2$. The resonance at $j = -1$ corresponds to the singular manifold $\chi = 0$ being arbitrary. If the $K(m+1, m, 1)$ models are Painelevé integrable, we require that the two resonance conditions $f_{2m} = f_{4m-2} = 0$ are satisfied identically, so that the other two functions ϕ_{2m} and ϕ_{4m-2} can be chose arbitrarily. With Kruskal's simplification, i.e. taking $\chi(\sigma, t) = \sigma + \psi(t)$, $\psi(t)$ is an arbitrary function of t, and ϕ_j are functions of t only. After a lengthy computation (with the help of Maple 7), one finds

$$\phi_0 = -(2m-2)(2m-1), \tag{25}$$

$$\phi_j = 0 \ (j \neq 0, 2m-2, 4m-4, 4m-3),$$

$$\phi_{2m-2} = -\frac{1}{m(2-2m)^{m-2}(2m-1)^{m-2}}(\psi_t + a),$$

$$\phi_{4m-4} = \frac{m^2-5m+3}{m^2(4m-3)(2m-2)^{2m-3}(2m-1)^{2m-3}} \cdot (\psi_t + a),$$

$$\phi_{4m-3} = -\frac{1}{m(2m-3)(2m-2)^{2m-3}(2m-1)^{2m-3}} \cdot \psi_{tt}.$$

Substitution of (25) into the resonance conditions implies that two resonance conditions are satisfied identically. Hence $K(m+1, m, 1)$ models are Painlevé integrable. Similarly one can investigate the integrability of the $K(m+1, m, m)$ models. Unfortunately we could not obtain a unified representation for the expansion coefficients ϕ_j because of complications, but for small m we can obtain their explicit expressions. For examples, when $m = 2$, we have the resonance points $j = -1, 6, 10$ and the expansion coefficients

$$\phi_0 = -20, \ \phi_j = 0 \ (j \neq 0, 2, 4, 8, 9),$$

$$\phi_2 = -\frac{5}{12}a, \ \phi_4 = \frac{1}{60}\psi_t - \frac{1}{192}a^2,$$

$$\phi_8 = -\frac{1}{216000}\psi_t^2 - \frac{1}{345600}a^2\psi_t - \frac{1}{2211840}a^4,$$

$$\phi_9 = -\frac{1}{72000}\psi_{tt}.$$

For $m = 3$, we have

$$\phi_0 = -42, \ \phi_j = 0 \ (j \neq 0, 2, 4, 6, 10, 12, 13),$$

$$\phi_2 = -\frac{7}{18}a, \ \phi_4 = -\frac{7}{3240}a^2,$$

$$\phi_6 = -\frac{1}{7056}\psi_t - \frac{1}{104976}a^3,$$

$$\phi_{10} = -\frac{1}{502951680}a^2\psi_t - \frac{31}{74826892800}a^5,$$

$$\phi_{12} = -\frac{1}{27183776256}\psi_t^2 - \frac{1}{202214009088}a^3\psi_t + \frac{3469}{3008449237248000}a^6,$$

$$\phi_{13} = -\frac{1}{2613824640}\psi_{tt}.$$

3.2. $K(m+2, m, n)$ $(n = 1, m)$ Models

The $K(m+2, m, 1)$ model we consider is of the form (3) with $n = m+2$. It reduces to the $K(m+2, m)$ model as $a = 0$, and it is Painlevé integrable [3]. For $m = 1$ it is the mKdV equation with the lower order term, and it is also Painlevé integrable. Substituting (24) with $\alpha = -1$ into the $K(m+2, m, 1)$ models, we obtain the recursion relation for the coefficients ϕ_j:

$$(j+1)(j-m)(j-2m+2)\phi_j = f_j(\chi_\sigma, \chi_t, \cdots, \phi_0, \phi_1, \phi_2, \cdots, \phi_{j-1}).$$

So resonances occur at $j = -1, m, 2m-2$. The arbitrary singularity manifold $\chi = 0$ corresponds to the resonance $j = -1$. Another two arbitrary functions ϕ_m and ϕ_{2m-2} will be introduced at the resonance points $j = m$ and $j = 2m-2$. The resonance conditions $f_m = f_{2m-2} = 0$ are satisfied identically. By Kruskal's simplification $\chi = \sigma + \psi(t)$ we obtain

$$\phi_0^2 = -(m-1)m, \phi_m = \frac{(m-1)m}{(m+1)\phi_0^{m+1}}(\psi_t + a), \phi_j = 0, j \neq 0, m.$$

Substituting these expressions into the resonance conditions, we see that two resonance conditions are satisfied automatically. So the $K(m+2, m, 1)$ models are Painlevé integrable. Similarly we can show that the $K(m+2,m,m)$ models are also Painlevé integrable.

4. Gauge Transformations

The concept of gauge equivalence between completely integrable models plays an important role in the theory of soliton [21, 22, 25]. For examples, it has been shown that the 1+1-dimensional continuous Heisenberg ferromagnetic model [25]

$$S_t = S \wedge S_{xx}$$

is gauge equivalent to the nonlinear Schrödinger equation

$$iq_t + q_{xx} + |q|^2 q = 0.$$

The mKdV equation is gauge equivalent to the WKI equation [22]. Its geometrical explanation is that, when the curvature of a plane curve satisfies the mKdV equation, its graph satisfies the WKI equation. So the natural question arises: what are the equations which are gauge equivalent to the $K(n,m,m)$ and $K(n,m,1)$ $(n = m+1, m+2)$ models?

4.1. K(m+2,m) Model

From Sect. 2 we know that the corresponding curve flow for the $K(m+2,m)$ model (11) with $a = 0$ is

$$\gamma_t = -(\kappa^m)_s \boldsymbol{n} - \frac{m}{m+1}\kappa^{m+1}\boldsymbol{t}.$$

With use of the graph $\gamma = (x, u(x,t))$, then $\boldsymbol{t} = (1, u_x)/\sqrt{1+u_x^2}$, $\boldsymbol{n} = (-u_x, 1)/\sqrt{1+u_x^2}$ and $\kappa = u_{xx}/(1+u_x^2)^{3/2}$. Using these expressions, the flow is written as the generalized WKI equation (5). Namely the $K(m+2,m)$ model is transformed to the generalized WKI equation by the transformations

$$\kappa = \frac{u_{xx}}{(1+u_x^2)^{3/2}},$$
$$s = \int_0^x (1+u_x^2)^{1/2}\,\mathrm{d}\,x,$$
$$t = t.$$

It has been shown by Konno et al. [26, 27] that the WKI equation (2) has N-loop solitons. In the next section we will show that the generalized WKI equation admits one-loop soliton solutions.

4.2. K(m+1,m) Model

From Sect. 2 we know that the $K(m+1,m)$ model arises from a centro-affine geometry. The corresponding curve flow is

$$\gamma_t = hf\boldsymbol{n} + (h^{-1}g - \kappa^{-1}fh_s)\boldsymbol{t},$$

where $f = -(\phi^m)_{\tilde{s}}$, $g = -2\phi^m$. With use of the graph $\gamma = (x, u(x,t))$, then $\phi = u_{xx}/(xu_x - u)^3$, $h = (xu_x - u)/\sqrt{1+u_x^2}$, and the flow reads

$$u_t + \left[\left(\frac{u_{xx}}{(xu_x - u)^3}\right)^m\right]_x = 0, \tag{26}$$

which is equivalent to the $K(m+1,m)$ model. The associated gauge transformations are

$$\phi = \frac{u_{xx}}{(xu_x - u)^3},$$
$$\tilde{s} = \int_0^x (xu_x - u)dx,$$
$$t = t.$$

In particular, the KdV equation is equivalent to (26) with $m = 1$, and the Harry Dym equation (18) is gauge equivalent to (26) with $m = -1/2$.

4.3. K(m+2,m,m) Model

It has been shown that the $K(m+2,m,m)$ model

$$\phi_t + \frac{m}{4(m+1)}(\phi^{m+2})_\sigma + (\phi^m)_{\sigma\sigma\sigma} + (\phi^m)_\sigma = 0 \tag{27}$$

arises from the motion of plane curves in the restricted conformal geometry [19]. The curve flow is

$$\gamma_t = -\frac{1}{4(m+1)}\phi^{m+1}\boldsymbol{T} - \frac{1}{2}(\phi^m)_\sigma \boldsymbol{N},$$

where $\boldsymbol{T} = \gamma_\sigma$ and $\boldsymbol{N} = \gamma_{\sigma\sigma}$ are, respectively, the tangent and normal in the restricted conformal geometry. $\sigma = \int_0^s \frac{2}{1+|\gamma|^2}\sqrt{1+u_x^2}dx$ is the arc-length. In the graph $(x, u(x,t))$ it is written as

$$u_t + \frac{1}{8}(1+x^2+u^2)^2\left[\left((1+x^2+u^2)\frac{u_{xx}}{(1+u_x^2)^{3/2}} + 2\frac{u - xu_x}{\sqrt{1+u_x^2}}\right)^m\right] = 0.$$

The associated gauge transformations are

$$\phi = (1 + x^2 + u^2)\frac{u_{xx}}{(1 + u_x^2)^{3/2}} + 2(u - xu_x),$$

$$\sigma = \int_0^x \frac{2}{1 + x^2 + u^2}(1 + u_x^2)^{1/2}dx,$$

$$t = t.$$

5. One-loop Solitons of the Generalized WKI Equation

It is known that the WKI equation (2) has N-loop soliton solutions [26, 27]. We now show that the generalized WKI equation (5) has one-loop soliton solutions. To this end we consider the motion of plane curves corresponding to the traveling wave of the $K(m+2, m)$-model.

Recall that the $K(m+2, m)$ model (11) with $a = 0$,

$$\kappa_t + (k^m)_{sss} + \frac{m}{m+1}(\kappa^{m+2})_s = 0, \tag{28}$$

is associated to the curve flow

$$\gamma_t = -(\kappa^m)_s \boldsymbol{n} - \frac{m}{m+1}\kappa^{m+1}\boldsymbol{t}. \tag{29}$$

In general, for any given curvature $\kappa(s,t)$, we define

$$\theta(s,t) = \int_0^s \kappa(\sigma, t)\mathrm{d}\sigma + \theta_0(t), \tag{30}$$

and

$$x(s,t) = \int_0^s \cos\theta(\sigma, t)\mathrm{d}\sigma + x_0(t),$$

$$u(s,t) = \int_0^s \sin\theta(\sigma, t)\mathrm{d}\sigma + y_0(t), \tag{31}$$

where $\theta_0(t)$, $x_0(t)$ and $y_0(t)$ are given functions of t. One can easily verify that for each fixed t, the curve $\gamma(s,t) = (x(s,t), u(s,t))$ is parametrized by the arclength, and its curvature is given by $\kappa(s,t)$. When $\kappa(s,t)$ solves (28), we can determine $\theta_0(t)$, $x_0(t)$ and $y_0(t)$ by substituting (30) and (31) into (29). We find

$$\theta_0(t) = -\int_0^t [(\kappa^m)_{ss}(0,\tau) + \frac{m}{m+1}\kappa^{m+2}(0,\tau)]\,\mathrm{d}\tau + \theta_0,$$

$$x_0(t) = \int_0^t [\sin\theta(0,\tau)(\kappa^m)_s(0,\tau) - \frac{m}{m+1}\cos\theta(0,\tau)\kappa^{m+1}(0,\tau)]\,\mathrm{d}\tau + x_0, \tag{32}$$

$$u_0(t) = \int_0^t [\cos\theta(0,\tau)(\kappa^m)_s(0,\tau) + \frac{m}{m+1}\sin\theta(0,\tau)\kappa^{m+1}(0,\tau)]\mathrm{d}\tau + u_0.$$

So (30)–(32) define a curve $\gamma = (x(x,t), u(s,t))$ which satisfies the $K(m+2, m)$-model flow (29). Here θ_0, x_0 and y_0 are arbitrary constants, reflecting the fact that the curvature determines the curve up to an Euclidean motion. In the following we choose $\theta_0 = x_0 = y_0 = 0$, so that the initial curve passes through the origin with the unit tangent $\vec{e}_1$.

Now suppose $\kappa = \hat{\kappa}(s - \lambda t)$ is the traveling wave of the $K(m+2, m)$ model and let $\hat{\gamma}$ be the curve determined by $\hat{\kappa}$

$$\hat{x}(s,t) = \int_0^s \cos\hat{\theta}(\sigma)\mathrm{d}\sigma$$

and

$$\hat{u}(s,t) = \int_0^s \sin\hat{\theta}(\sigma)\mathrm{d}\sigma,$$

where

$$\hat{\theta} = \int_0^s \hat{\kappa}(\sigma)\mathrm{d}\sigma.$$

If the curvature does not vanish, we can use the angle θ to parametrize $\hat{\gamma}$. Setting $w = \kappa^{m+1}$, w satisfies

$$w_{\theta\theta} + w = \lambda_0 + c_0 w^{-\frac{1}{m+1}}, \tag{33}$$

where λ_0 and c_0 are constants, $\lambda_0 = (m+1)\lambda/m$. We consider two cases:

Case 1. $m = -2$. The solutions of (33) are given by

$$w = \begin{cases} \frac{1}{2}\lambda_0\theta^2 + d, & \text{if } 1 - c_0 = 0, \\ d\cos a\theta + \frac{\lambda_0}{a^2}, & \text{if } 1 - c_0 = a^2 > 0, \\ -d\cosh a\theta - \frac{\lambda_0}{a^2}, & \text{if } 1 - c_0 = -a^2 < 0. \end{cases} \tag{34}$$

where d is an arbitrary constant. Corresponding to each line of (34), we obtain the following solutions:

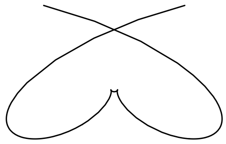

Fig. 1. A traveling wave (35) with $\lambda_0 = 2$ and $d = -1$ to the generalized WKI equation with $m = -2$.

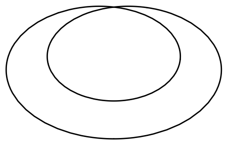

Fig. 2. A closed traveling wave solution (36) with $a = 1/2$ and $d = -\lambda_0 = -1$ to the generalized WKI equation with $m = -2$.

1.1. $1 - c_0 = 0$.

$$x(\theta, t) = \frac{1}{2}\lambda_0(\theta^2 \sin\theta + 2\theta\cos\theta - 2\sin\theta) + d\sin\theta,$$

$$u(\theta, t) = \frac{1}{2}\lambda_0(-\theta^2 \cos\theta + 2\theta\sin\theta + 2\cos\theta) - d\cos\theta. \tag{35}$$

A curve for $\lambda_0 = 2$, $d = -1$ is plotted in Figure 1.

1.2. $1 - c_0 = a^2 > 0$, $c_0 \neq 0$.

$$\begin{aligned} x(\theta, t) &= \frac{d}{2(a+1)}\sin[(a+1)\theta] \\ &\quad + \frac{d}{2(a-1)}\sin[(a-1)\theta] + \frac{\lambda_0}{a^2}\sin\theta, \\ u(\theta, t) &= \frac{d}{2(a-1)}\cos[(a-1)\theta] \\ &\quad - \frac{d}{2(a+1)}\cos[(a+1)\theta] - \frac{\lambda_0}{a^2}\cos\theta, \end{aligned} \tag{36}$$

which gives a closed curve. Figure 2 shows a curve corresponding to $a = 1/2$, $d = -\lambda_0 = -1$.

1.3. $1 - c_0 = a^2 > 0$, $c_0 = 0$.

$$x(\theta, t) = \frac{d}{4}\sin 2\theta + \frac{d}{2}\theta + \lambda_0 \sin\theta,$$

$$u(\theta, t) = -\frac{d}{4}\cos 2\theta - \lambda_0\cos\theta. \tag{37}$$

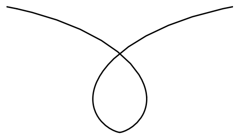

Fig. 3. One-loop soliton (37) with $\lambda_0 = -d = 1$ to the generalized WKI equation with $m = -2$.

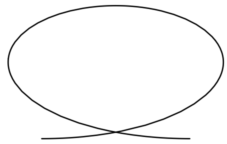

Fig. 4. One-loop soliton (38) with $a = \sqrt{2}/2$ and $\lambda_0 = d = 1$ to the generalized WKI equation with $m = -2$.

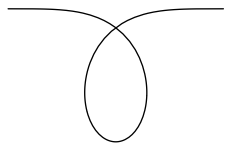

Fig. 5. One-loop soliton corresponding to (39) of the generalized WKI equation with $m = 2$.

A curve for $d = -1$ and $\lambda_0 = 1$ is plotted in Figure 3.

1.4. $1 - c_0 = -a^2 < 0$.

$$\begin{aligned} x(\theta, t) &= -\frac{d}{c_0}\cosh a\theta \sin\theta \\ &\quad - \frac{ad}{c_0}\sinh a\theta\cos\theta - \frac{\lambda_0}{a^2}\sin\theta, \\ u(\theta, t) &= -\frac{ad}{c_0}\sinh a\theta\sin\theta \\ &\quad + \frac{d}{c_0}\cosh a\theta\cos\theta + \frac{\lambda_0}{a^2}\cos\theta. \end{aligned} \tag{38}$$

Figure 4 shows a curve corresponding to $a = \sqrt{2}/2$, $\lambda_0 = 1$ and $d = 1$.

Case 2. $m \neq -2$, $c_0 = 0$. We have

$$w = \lambda_0 + d\cos\theta. \tag{39}$$

A curve corresponding to $m = 2$, $\lambda_0 = d = 3/2$ is plotted in Figure 5.

6. Conclusions

In this paper we have shown that the nonlinear dispersive $K(m+2, m)$ and $K(m+1, m)$ models arise naturally from the motions of plane curves in the Euclidean and centro-affine geometries. The $K(n,m,1)$ and $K(n,m,m)$ ($n = m+1, m+2$) models, as the generalizations to the $K(n,m)$ models, have been introduced by considering motions of curves in other geometries. Such derivation implies that these models are kinematically integrable, but no parameter appeared in the integrability conditions of linear problems. It has also been shown that those models are Painlevé integrable.

Several integrable equations have been derived via the equivalence of the curvature and graph of the curves. In view of this equivalence, the mKdV equation is gauge equivalent to the WKI equation since they come respectively from the AKNS-scheme and the WKI-scheme, and the $K(m+2,m)$ model is equivalent to the generalized WKI equation. It is interesting to note that the generalized WKI equation also admits one-loop soliton solution. This property has been known for a long time for the WKI equation [26, 27]. In the same way we can find from the motion of curves in the similarity geometry that the third-order Burgers equation

$$\phi_t = \phi_{\sigma\sigma\sigma} - 3(\phi\phi_\sigma)_\sigma + 3\phi^2\phi_\sigma$$

is gauge equivalent to the equation [18]

$$u_t + [\frac{u_{xx}}{(1+u_x^2)^{\frac{3}{2}}}]^{-3}[\frac{u_{xx}}{(1+u_x^2)^{\frac{3}{2}}}]_x = 0,$$

and in the affine geometry, the Sawada-Kotera equation

$$\phi_t + \phi_5 + 5\phi\phi_{\sigma\sigma\sigma} + 5\phi_\sigma\phi_{\sigma\sigma} + 5\phi^2\phi_\sigma = 0$$

is equivalent to the affine Sawada-Kotera equation [18]

$$u_t + [u_{xx}^{-5/3}u_{xxxxx} - \frac{5}{3}u_{xx}^{-8/3}u_{xxx}^2]_x = 0.$$

Finally, it is interesting to investigate whether these models are integrable in other meanings and to find physical applications for these models.

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